Journal of Statistical Physics, Vol. 123, No. 6, June 2006 (© 2006)
DOI: 10.1007/s10955-006-9128-5

Thermodynamic Limit for a Spin Lattice

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Received June 10, 2005; accepted May 3, 2006
Published Online: June 27, 2006

An integrable spin lattice is a higher dimensional generalization of integrable spin chains. In this paper we consider a special spin lattice related to quantum mechanical interpretation of the three-dimensional lattice model in statistical mechanics (Zamolodchikov and Baxter). The integrability means the existence of a set of mutually commuting operators expressed in the terms of local spin variables. The significant difference between spin chain and spin lattice is that the commuting set for the latter is produced by a transfer matrix with two equitable spectral parameters. There is a specific bilinear functional equation for the eigenvalues of this transfer matrix.

The spin lattice is investigated in this paper in the limit when both sizes of the lattice tend to infinity. The limiting form of bilinear equation is derived. It allows to analyze the distributions of eigenvalues of the whole commuting set. The ground state distribution is obtained explicitly. A structure of excited states is discussed.

KEY WORDS: Three-dimensional integrable spin systems, Zamolodchikov-Baxter model.

1. INTRODUCTION

The integrability of Zamolodchikov-Baxter three-dimensional lattice model[1,2] is based on the commutativity of transfer-matrices $T(\theta_1, \theta_2, \theta_3)$,

$$[T(\theta_1, \theta_2, \theta_3), T(\theta_1, \theta_2', \theta_3')] = 0, \tag{1}$$

where θ_j are Zamolodchikov dihedral angles. We understand T as an operator in the vertex formulation[3] of Zamolodchikov-Bazhanov-Baxter model[4] with two spin states. Matrix T represents graphically a *layer* of three-dimensional R-matrices.[5] Let the sizes of the layer be $N \times M$, so that the transfer matrix is the quantum

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mechanical operator acting in the Hilbert space $\mathcal{H} = h^{\otimes NM}$, where $h = \mathbb{C}^2$ is the two-dimensional state space for local spin variable, and $\dim \mathcal{H} = 2^{NM}$.

Two parameters θ_2 and θ_3 are varied in Eq. (1), it reveals the three-dimensional nature of the transfer matrix T. Relation (1) implies the existence of a discrete set of commutative operators $\{t_{m,n}(\theta_1)\}$,

$$T(\theta_1, \theta_2, \theta_3) = \sum_{m,n} t_{m,n}(\theta_1) G_{m,n}(\theta_2, \theta_3), \tag{2}$$

where $G_{m,n}(\theta_2, \theta_3)$ are some numerical coefficients. The problem of diagonalization of $T(\theta_1, \theta_2, \theta_3)$ for any θ_2, θ_3 is equivalent to the problem of simultaneous diagonalization of $\{t_{m,n}(\theta_1)\}$. In what follows, we consider a specific complete set of $\{t_{m,n}\}$ which can be defined with the help of auxiliary problems.

It is well known, the Zamolodchikov model and its generalization—Bazhanov-Baxter model[(4)]—are related to the generalized chiral Potts model.[(6)] The prescription for the derivation of the desired set $\{t_{m,n}\}$ may be formulated in the quantum group terms. Let $L(u)$ be a Lax operator for $\mathcal{U}_q(\widehat{gl}_N)$ corresponding to the minimal cyclic representation in the quantum space and N-dimensional vector representation in the auxiliary space.[(6,7)] Let further $\mathbb{T}(u)$ be the monodromy of $L(u)$ for the chain of the length M, $\mathbb{T}(u) = L_1(u)L_2(u)\ldots L_M(u)$. The complete set of integrals of motion is generated by all quantum characters of $\mathbb{T}(u)$. The following expression gives a scheme for the definition of our set $\{t_{m,n}\}$:

$$\text{“q-det”}[\phi(u)v - \mathbb{T}(u)] \;=\; \sum_{n=0}^{N}\sum_{m=0}^{M} (-)^{nm+n+m} u^m v^n t_{m,n}, \tag{3}$$

where $\phi(u)$ is some diagonal matrix making the “q-characteristic polynomial” self-consistent. Here N and M are exactly the sizes of the layer. Alternatively, $t_n(u) = \sum_{m=0}^{M} u^m t_{m,n}$ is the transfer-matrix for the length-M chain of $\mathcal{U}_q(\widehat{gl}_N)$ Lax operators corresponding to the minimal cyclic representation in the quantum space and to mth fundamental representation (rank-m antisymmetric tensors) in the auxiliary space. Note, one should consider the minimal cyclic representation with $q^2 = 1$ and only one independent parameter corresponding to the single θ_1. Arbitrary parameters of cyclic representation correspond to an inhomogeneous set of $\{\theta_1\}$.

Another combinatorial scheme producing the same $\{t_{m,n}\}$ was proposed in Refs. 8 and 9. The combinatorial formulation uses the natural algebra of observables of the quantum-mechanical system—the set of NM local Pauli matrices associated with the local quantum spaces $h = \mathbb{C}^2$ of the layer-to-layer transfer matrix $T(\theta_1, \theta_2, \theta_3)$. All $t_{m,n}$ are simple polynomials in the algebra of observables. This scheme is invariant from the point of view of $2 + 1$ dimensional integrability, in particular its rank-size $N \leftrightarrow M$ duality is evident. The detailed combinatorial formulation will be given in the first section. Since this framework implies the local algebra of observables associated with the vertices of the layer-lattice, we call it "spin lattice." It is important to note, all $t_{m,n}$ are Hermitian, i.e. the model is indeed a model of *quantum mechanics*.

The eigenvalues of the set $\{t_{m,n}\}$ may be found as a solution of a system of bilinear equations. In the language of auxiliary transfer matrices for the generalized chiral Potts model, the system of bilinear equations is the complete set of fusion relations for fundamental transfer matrices $t_n(u)$. The reader may find the investigation and discussion of the fusion relations and Bethe Ansatz for Zamolodchikov model for $N = 3$ in Refs. 4, 10 and 11. In the direct $3D$ scheme the whole system of fusion relations is encoded into a single spectral equation.[8,12]

The problem is to find the simultaneous eigenvalues of all $t_{m,n}$. One way to solve this problem is the nested Bethe Ansatz equations for the fusion algebra. The crucial point in the Bethe Ansatz theory is the limit when the Bethe roots form a continuous distribution: length of the chain $M \to \infty$, rank of the symmetry group $N - 1$ stays finite. Even if one sends now $N \to \infty$, the resulting theory will correspond to the singular aspect ratio $\frac{N}{M} \to 0$. The Bethe Ansatz approach may give a correct answer only for a quantity independent on N/M.

Contrary to this, the spectral equation in the direct $3D$ scheme is initially $N \leftrightarrow M$ invariant. In this paper the spectral equation is evaluated in the limit

$$N, M \to \infty, \quad \frac{N}{M} \to \zeta \tag{4}$$

where ζ in the non-singular aspect ratio of the layer lattice. The main result of this paper is the exact distribution $f(m, n; \theta_1, \zeta)$ of the largest eigenvalues (the ground state), $t_{m,n} = \text{const}\sqrt{NM}e^{NMg(\theta_1)/2} f(m, n; \theta_1, \zeta)$ in the limit (4). The other result is the limiting form of the spectral equation allowing one to describe (at least qualitatively) the gap-less excitations above the ground state.

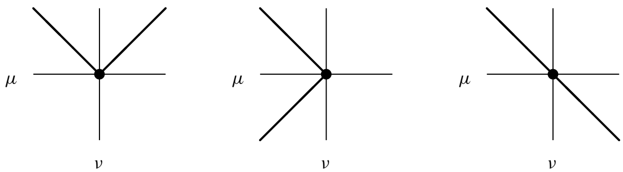
This paper is organized as follows. In Secs. 1–3, we formulate first the framework of the spin lattice, recall its finite $N \times M$—volume spectral equation and make its leading term evaluation. Content of the first three sections is a repetition of Refs. 8, 9, 12–14. Next, in the fourth section, we expose some preliminary numerical results for the spectrum of $t_{m,n}$ and discuss the main idea for the limiting (4) procedure. In the fifth section we re-write the spectral equations in the thermodynamic limit $N, M \to \infty$. In the sixth section the qualitative analysis of the thermodynamical spectral equation is given, the distribution of the ground state eigenvalues of $t_{m,n}$ is obtained, and the structure of excitations is discussed.

2. FORMULATION OF THE SPIN LATTICE SYSTEM

All the ways to produce the set $\{t_{m,n}\}$, either via Lax operators for cyclic representation of $\mathcal{U}_{q=-1}(\widehat{gl}_N)$ or via $3D$ linear problem,[8] finally may be reformulated in the following combinatorial form.

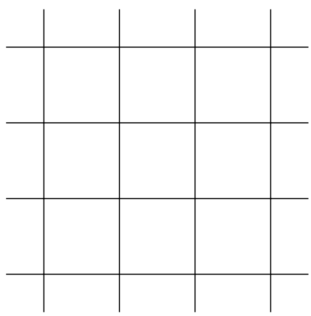
Consider a square lattice with the size $N \times M$ and periodical boundary conditions in both directions—exactly the layer of (1). Each vertex j of the lattice may be labelled by the pair of the indices $j = (\nu, \mu)$, $\nu \in \mathbb{Z}_N$, $\mu \in \mathbb{Z}_M$. A local triplet of the Pauli matrices σ_j^x, σ_j^y and $\sigma_j^z = \mathrm{i}\,\sigma_j^x \sigma_j^y$ is assigned to each vertex.

Consider a set of non-self-intersecting paths on the periodic lattice with the following rules of bypassing a vertex and following factors γ_j associated with each variant of bypassing (note the multiplier κ in the third variant):



$\gamma_j = \sigma^x_{\nu,\mu}$ $\qquad$ $\gamma_j = \sigma^y_{\nu,\mu}$ $\qquad$ $\gamma_j = \kappa\sigma^z_{\nu,\mu}$

An example of such path for 4×4 lattice is drawn below:



Any path $\mathcal{P}$ on the torus has a homotopy class $c(\mathcal{P}) = m\mathcal{A} + n\mathcal{B}$, where $\mathcal{A}$ is the cycle from left to right and $\mathcal{B}$ is the cycle from bottom to top. In the other words, m is the horizontal winding number and n is the vertical winding number of the path $\mathcal{P}$. The path in the example above has $n = m = 1$.

For fixed winding numbers n and m let

$$J_{m,n}(\kappa) = \sum_{\mathcal{P}\,:\,c(\mathcal{P})=m\mathcal{A}+n\mathcal{B}} \prod_{\text{along } \mathcal{P}} \gamma_j \tag{5}$$

be the sum of the products $\prod_{\text{along}\mathcal{P}} \gamma_j$ of γ-factors along a path $\mathcal{P}$ for all possible paths with the given winding numbers. The empty path gives $J_{0,0} = 1$. The winding numbers of $J_{m,n}$ take the values $m = 0, 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots, N$.

It is known,[8,9] operators $J_{m,n}(\kappa)$ obey the following exchange relations:

$$J_{m,n}(\kappa)J_{m',n'}(\kappa) \;=\; (-)^{nm'+n'm}\; J_{m',n'}(\kappa)J_{m,n}(\kappa). \tag{6}$$

These relations mean that all $J_{m,n}$ may be quasi-diagonalized simultaneously: there exists a basis $|\psi_t, j\rangle$ in the Hilbert space such that

$$J_{m,n}\;|\psi_t, j\rangle \;=\; \sum_{k=1,2} |\psi_t, k\rangle[(\sigma^x)^m(\sigma^y)^n]_{kj}\; \mathrm{i}^{nm}\; t_{m,n}, \tag{7}$$

where σ^x and σ^y are 2×2 standard Pauli matrices. In the basis-independent form,

$$\begin{aligned} J_{m,n}(\kappa) \;&=\; \mathrm{i}^{nm}\,(\sigma^x)^m(\sigma^y)^n t_{m,n}(\kappa),\\ [\sigma^x, t_{m,n}] = [\sigma^y, t_{m,n}] = [t_{m,n}, t_{m',n'}] \;&=\; 0,\\ \sigma^x\sigma^y = -\sigma^y\sigma^x\;, \quad (\sigma^x)^2 = (\sigma^y)^2 &= 1. \end{aligned} \tag{8}$$

The Ansatz (8) "solves" the exchange relations (6). Operators σ^x and σ^y may be expressed in the terms of the algebra of observables. Without loss of generality we may choose

$$\sigma^x \;=\; \prod_\nu \sigma^x_{\nu,1} \quad \text{and} \quad \sigma^y \;=\; \prod_\mu \sigma^y_{1,\mu}. \tag{9}$$

The possibility of such choice follows from a detailed inspection of $J_{m,0}$ and $J_{0,n}$.

In the generalized chiral Potts model scheme, σ^x is related to the global $U(1)$-charge of $(\mathcal{U}_q(\widehat{gl}_N))^{\otimes M}$, and σ^y comes from a dynamical Yang-Baxter equation.

The set of $J_{m,n}$ (and $\{t_{m,n}\}$ as well) is the set of "integrals of motion" for Zamolodchikov model[1] in its vertex formulation.[3] Namely,[8,9,13] the layer-to-layer transfer matrix of Zamolodchikov model $T(\theta_1, \theta_2, \theta_3)$ commutes with all $J_{m,n}(\kappa)$ for $\kappa \equiv \tan\frac{\theta_1}{2}$ and arbitrary θ_2, θ_3. In this paper we prefer to call the commuting operators $t_{m,n}$ the *moduli* since in the classical limit they become the moduli of the classical spectral curve.[15]

The advantage of the present quantum-mechanical formulation is that if κ is real, all $J_{m,n}$ and $t_{m,n}$ are self-adjoint operators since the Pauli matrices are self-adjoint, therefore the model is evidently physical. An eigenstate of the model is defined by eigenvalues of all $t_{m,n}$—one can label the eigenstates by the corresponding values of $\{t_{m,n}\}$. Our aim is to describe all eigenstates.

3. FINITE SIZE SPECTRAL EQUATIONS

In this section we recall the functional equation for the set of $t_{m,n}$. For its rigorous derivation see. Ref. 12.

Consider the following generating function:

$$J(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N}(-)^{n+m+nm}x^m y^n J_{m,n} \tag{10}$$

(confer Eq. (3)), where x and y are arbitrary parameters. In the basis of the auxiliary σ-matrices (8) $J(x,y)$ is

$$J(x,y) = t_{0,0}(x,y) - \sigma^x t_{1,0}(x,y) - \sigma^y t_{0,1}(x,y) - \sigma^z t_{1,1}(x,y) \tag{11}$$

where

$$\begin{aligned} t_{0,0}(x,y) &= \sum_{m,n} x^{2m}y^{2n}t_{2m,2n},\\ t_{1,0}(x,y) &= \sum_{m,n}(-)^n x^{2m+1}y^{2n}t_{2m+1,2n},\\ t_{0,1}(x,y) &= \sum_{m,n}(-)^n x^{2m}y^{2n+1}t_{2m,2n+1},\\ t_{1,1}(x,y) &= \sum_{m,n}(-)^{m+n}x^{2m+1}y^{2n+1}t_{2m+1,2n+1}. \end{aligned} \tag{12}$$

It is known,[9,12,13] the complete Abelian algebra of $t_{m,n}$ is generated by the polynomial decomposition of

$$t_{0,0}(x,y)^2 - t_{1,0}(x,y)^2 - t_{0,1}(x,y)^2 - t_{1,1}(x,y)^2 = F(x^2,y^2), \tag{13}$$

where

$$F(\lambda^N,\mu^N) = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1}(1-\lambda e^{2\pi i n/N} - \mu e^{2\pi i m/M} - \kappa^2\lambda\mu e^{2\pi i(n/N+m/M)}), \tag{14}$$

is a polynomial of $x^2 = \lambda^N$ and $y^2 = \mu^N$:

$$F(x^2,y^2) = \sum_{P=0}^{M}\sum_{Q=0}^{N} x^{2P}y^{2Q}F_{P,Q}. \tag{15}$$

As it was mentioned in the introduction, Eq. (13) encodes the whole fusion algebra of auxiliary transfer matrices for $\mathcal{U}_{q=-1}(\widehat{gl}_N)$, the reader may find some explanation for $N=3$ in the Appendix.

The right hand side of (13) may be re-written as

$$\sum_{P,Q} x^{2P}y^{2Q}\sum_{m,n}(-)^{m+n+mn+mQ+nP}t_{m,n}t_{2P-m,2Q-n}. \tag{16}$$

Equation (13) is the principal solution of the model, in the same way as the Bethe Ansatz is the principal solution for the spin chains: the problem of diagonalization of $2^{NM} \times 2^{NM}$ matrices $t_{m,n}$ is reduced to a system of $(N+1)(M+1)$ algebraic equations.

$$\sum_{m,n}\sum(-)^{m+n+mn+mQ+nP} t_{m,n} t_{2P-m,2Q-n} = F_{P,Q}. \tag{17}$$

4. THE LEADING TERM

Suppose, there are no zero terms in the product (14). Then F in (13) is exponentially big, and one may definitely conclude,[14]

$$\text{Each of } \left(t_{\alpha,\beta}(x,y)\right)^2_{\alpha,\beta=0,1} \sim |F(x^2,y^2)| \sim \mathrm{e}^{NM\mathfrak{g}(\lambda,\mu;\kappa^2)}, \tag{18}$$

where $x = \lambda^{N/2}$, $y = \mu^{M/2}$, and the integral

$$\mathfrak{g}(\lambda,\mu;\kappa^2) = \lim_{N,M\to\infty} \frac{1}{NM} \log|F(X,Y)|$$

$$= \frac{1}{(2\pi)^2}\int\int_0^{2\pi} d\phi d\phi' \log|1 - \lambda \mathrm{e}^{\mathrm{i}\phi} - \mu \mathrm{e}^{\mathrm{i}\phi'} - \kappa^2\lambda\mu \mathrm{e}^{\mathrm{i}(\phi+\phi')}|, \tag{19}$$

being parameterized by

$$|\lambda| = \frac{\sin r_2}{\sin r_1}, \quad |\mu| = \frac{\sin r_3}{\sin r_1}, \quad \kappa^2 = \frac{\sin r_0 \sin r_1}{\sin r_2 \sin r_3}, \tag{20}$$

with r_j bounded by

$$r_0 + r_1 + r_2 + r_3 = \pi \quad \text{and} \quad 0 \le r_1 + r_2,\ r_1 + r_3,\ r_2 + r_3 < \pi, \tag{21}$$

has the value[14]

$$\mathfrak{g}(\lambda,\mu;\kappa^2) = -\log 2\sin r_1 + \sum_{j=0}^{3}\left(\frac{r_j}{\pi}\log 2\sin r_j + \Phi(r_j)\right), \tag{22}$$

where $\Phi(r)$ is the polylogarithm[2]:

$$\Phi(r) = \sum_{m=1}^{\infty} \frac{\sin(2mr)}{2\pi m^2}. \tag{23}$$

This value is closely related to Baxter's result for the bulk free energy of the Zamolodchikov model.[2,13,14]

Relation (18) gives the common bulk term for $t_{\alpha,\beta}(x,y)$ on all eigenstates. Note, (22) corresponds to the asymptotically infinite values of the spectral parameters $x = \lambda^{N/2}$ and $y = \mu^{M/2}$. The answer (22) is useless from the quantum mechanical point of view since the bulk term does not take into account the structure of $t_{m,n}$. We need the finite size corrections.

5. NUMERICAL EVALUATION FOR FINITE N, M

We started the investigation of (17) with the numerical tests for relatively small N, M (up to $N = M = 8$) and for simple choices of κ.

The principal observation for finite N, M is the following. Excluding $t_{m,n}$ from (17) step-by-step, one comes to a final polynomial equation for a single $t_{m,n}$: such polynomial equation is exactly the characteristic equation for the operator $t_{m,n}$. Therefore, the system (17) and the problem of direct diagonalization of operators $t_{m,n}$ are equivalent. In other words, any solution of Eq. (17) is indeed an eigenstate. For this reason we call (17) the complete Abelian algebra.

Note in addition the parity property: if a set $\{t_{m,n}\}$ solves the Eq. (17), then the set $\{\tilde{t}_{m,n}\}$,

$$\tilde{t}_{2m+\alpha,2n+\beta} = \varepsilon_{\alpha,\beta} t_{2m+\alpha,2n+\beta}, \tag{24}$$

where $\alpha, \beta = 0, 1$ and $\varepsilon_{\alpha,\beta}$ are four arbitrary signs, solves (17) as well. This ambiguity corresponds to the ambiguity of definition of the auxiliary σ^x, σ^y, σ^z.

It is useful to visualize the domain of the indices of $\{t_{m,n}\}$ as the set Π of points (m, n) on the "momentum" plane:

$$\Pi \;=\; \{(m, n)\} 0 \le n \le N\ , \quad 0 \le m \le M. \tag{25}$$

The domain of $F_{P,Q}$ is the same, it is the Newton polygon for $F(x^2, y^2)$. On the boundary of the rectangular Π the eigenvalues of $t_{m,n}$ as well as the values of $F_{P,Q}$ are simple. Just putting e.g. $y = 0$ in (11), one gets

$$t_{0,0}(x, 0)^2 - t_{1,0}(x, 0)^2 = (1 - x^2)^M. \tag{26}$$

This equation defines all possible boundary eigenvalues $t_{m,0}$. Subject of interest is the calculation of $t_{m,n}$ in the middle of Π.

Numerical calculations show that for all eigenstates the absolute values of $t_{m,n}$ as well as the coefficients $F_{m,n}$ grow significantly when (m, n) goes from the boundary of Π to its middle. One eigenstate (up to the parity equivalence (24)) is strictly separated from all others: absolute value of any its $t_{m,n}$ is the maximal with respect to values of the same $t_{m,n}$ for all other eigenstates. We will call it the ground state.

For a given eigenstate, especially for the ground state, the values of $t_{m,n}$ are maximal in some point $(m, n) = (P_0, Q_0)$ in the middle of rectangular Π. In the same point the coefficient F_{P_0,Q_0} has the maximal absolute value with respect to all other $F_{P,Q}$. The observed feature of the ground state is that the value of $\frac{t_{P_0+m,Q_0+n}}{t_{P_0,Q_0}}$, where $|m| \ll M$ and $|n| \ll N$, depends essentially only on N/M and κ. The same asymptotical independence of N, M is valid for $\frac{F_{P_0+m,Q_0+n}}{F_{P_0,Q_0}}$ as well. Expression (19) is the result of a competition between the domain of maximal values of $F_{P,Q}$ and big (or small) values of $\lambda^{MP}\mu^{NQ}$ accompanying $F_{P,Q}$, see Eq. (15).

Another feature of $\{t_{m,n}\}$ may be mentioned. We observed that the sets of signs of $\{t_{m,n}\}$ for (m, n) surrounding (P_0, Q_0) are different (up to (24)) for different eigenstates.

These observations allow us to suggest an idea for evaluation of (17). Since both $\{t_{m,n}\}$ and $\{F_{P,Q}\}$ have a domain of dominance—the neighborhood of (P_0, Q_0) in the middle of Π—we can stand to the point (P_0, Q_0) and concentrate on its neighborhood. The boundary of Π is far from (P_0, Q_0), and in the limit $N, M \to \infty$ the boundary goes to infinity, so that the domain of the dominance becomes the open $\mathbb{Z}^2$. For finite N, M, in the neighborhood of (P_0, Q_0)

$$|t_{P_0+m,Q_0+n}|^2 \sim |F_{P_0+m,Q_0+n}| \sim \mathrm{e}^{NM\mathfrak{g}(1,1;\kappa^2)}, \ |m| \ll M \ , \quad |n| \ll N \ , \tag{27}$$

so that the bulk exponent is just the common factor for all eigenstates. Cancelling it, one does can evaluate the spectral Eq. (17) in the domain of dominance in the limit (4). This will be done in the next section.

6. SPECTRAL EQUATION IN THE THERMODYNAMIC LIMIT

To rewrite Eqs. (13) or (17) in the thermodynamic limit $N, M \to \infty$ with $\frac{N}{M} \to \zeta$, we need to introduce several notations.

Define parameters c and a via

$$c \;=\; \cot\frac{a}{2} \;=\; \sqrt{\frac{1+\kappa^2}{3-\kappa^2}} \;\iff\; \kappa^2 \;=\; \frac{\sin\frac{3a}{2}}{\sin\frac{a}{2}}. \tag{28}$$

At $N, M \to \infty$ the middle point (P_0, Q_0) is defined by

$$M\left(1-\frac{a}{\pi}\right) \;=\; P_0 - u_1 \ , \quad N\left(1-\frac{a}{\pi}\right) \;=\; Q_0 - u_2 \tag{29}$$

where P_0 and Q_0 are *even* integers while u_1 and u_2, $-1 < u_1, u_2 \le 1$, are fractional parts. If a is not a rational fraction of π, both u_1 and u_2 are extra variables.

Define next the quadratic form parameterized in the terms of c and aspect ratio ζ:

$$\Omega(p,q) \;=\; \frac{\pi}{2}\left(\zeta\frac{1+c^2}{2c}p^2 + \frac{1-c^2}{c}pq + \zeta^{-1}\frac{1+c^2}{2c}q^2\right). \tag{30}$$

The $N, M \to \infty$ limit of (17) is based on the following asymptotic of the coefficients $F_{P,Q}$:

$$F_{P_0+p,Q_0+q} = (-)^{p+q+pq}\, \mathrm{e}^{NM\mathfrak{g}_0(\kappa^2)} \cdot \mathrm{e}^{-\Omega(p+u_1,q+u_2)} \cdot f_0 \times \left(1 + \frac{f_1 + f_2\Omega(p+u_1, q+u_2)}{NM} + \cdots\right), \tag{31}$$

where

$$\mathfrak{g}_0(\kappa^2) \equiv \mathfrak{g}(1, 1; \kappa^2) = \left(1 - \frac{3a}{2\pi}\right)\log\kappa^2 + 3\Phi\left(\frac{a}{2}\right) - \Phi\left(\frac{3a}{2}\right). \tag{32}$$

Coefficients f_0, f_1, f_2 are some functions of κ^2, ζ, u_1 and u_2 (a sketch derivation of (31) and the value of f_0 is given in the Appendix).

Define $\tau_{m,n}$ as the fine structure of $t_{m,n}$,

$$\tau_{m,n} = \frac{t_{P_0+m,Q_0+n}}{\sqrt{f_0}\,\mathrm{e}^{\frac{1}{2}NM\mathfrak{g}_0(\kappa^2)}}. \tag{33}$$

Here, according to the idea of the previous section, we have moved to the middle point (P_0, Q_0) of the domain Π and canceled common bulk factor. Substituting (31) and (33) into (17), cancelling the exponents and taking the limit $N, M \to \infty$, we come to the following equations for $\tau_{m,n}$,

$$\sum\sum_{m,n\in\mathbb{Z}} (-)^{m+n+mn}\tau_{p+m,q+n}\tau_{p-m,q-n} = \mathrm{e}^{-\Omega(p+u_1,q+u_2)}. \tag{34}$$

The next substitution

$$\tau_{m,n} = c_{m,n}\,\mathrm{e}^{-\frac{1}{2}\Omega(m+u_1,n+u_2)} \tag{35}$$

transforms (34) into the free from u_1, u_2 form:

$$\sum\sum_{m,n\in\mathbb{Z}} (-)^{m+n+mn}\mathrm{e}^{-\Omega(n,m)}c_{p-m,q-n}c_{p+m,q+n} = 1 \quad \forall\ p, q \in \mathbb{Z}. \tag{36}$$

Equations (34) and (36) are two forms of (13) in the thermodynamical limit.

7. ANALYSIS OF (36)

For the analysis of (36), let us modify it slightly at the first:

$$\sum\sum_{m,n\in\mathbb{Z}} (-)^{m+n+mn}\mathrm{e}^{-\beta\Omega(n,m)}c_{p-m,q-n}c_{p+m,q+n} = 1 \quad \forall\ p, q \in \mathbb{Z}, \tag{37}$$

where the cut-off parameter $\beta \geq 1$.

Consider for a moment $\zeta = 1$. In this case

$$\Omega(m, n) = \frac{\pi}{4}(c^{-1}(m+n)^2 + c(m-n)^2), \tag{38}$$

and we have two small parameters in Eq. (37),

$$Q = \mathrm{e}^{-\beta\pi/4c} \quad \text{and} \quad \tilde{Q} = \mathrm{e}^{-\beta\pi c/4}. \tag{39}$$

Equation (37) may be analyzed in the terms of the perturbative expansion with respect to Q, $\tilde{Q}$. The zero order reads

$$c_{p,q}^2 + o(1) = 1 \quad \Rightarrow \quad c_{p,q} = \varepsilon_{p,q}\,(1 + o(1)), \tag{40}$$

where $\varepsilon_{p,q} = (\pm)$ is the sign of $c_{p,q}$. In the first non-trivial order,

$$c_{p,q} = \varepsilon_{p,q}\,(1 + (\varepsilon_{p+1,q}\varepsilon_{p-1,q} + \varepsilon_{p,q-1}\varepsilon_{p,q+1})Q\tilde{Q} + \cdots). \tag{41}$$

This procedure may be continued, the result is a series with respect to Q and $\tilde{Q}$,

$$c_{p,q} = \varepsilon_{p,q}\left(1 + \sum_{m,n>0} \chi_{p,q}^{(m,n)} Q^m \tilde{Q}^n\right), \tag{42}$$

where coefficients $\chi_{p,q}^{(m,n)}$ are sums of products of $\varepsilon_{m,n}$ for (m, n) surrounding (p, q): The first few nonzero $\chi_{p,q}^{(m,n)}$ with $m + n \leq 4$ are

$$\chi_{p,q}^{(1,1)} = \varepsilon_{p+1,q}\varepsilon_{p-1,q} + \varepsilon_{p,q-1}\varepsilon_{p,q+1}, \tag{43}$$

$$\begin{aligned}\chi_{p,q}^{(2,2)} = {} & \varepsilon_{p,q+1}\varepsilon_{p,q-1}\varepsilon_{p+1,q-1}\varepsilon_{p-1,q-1} \\ & + \varepsilon_{p+1,q}\varepsilon_{p-1,q}\varepsilon_{p-1,q+1}\varepsilon_{p-1,q-1} + \varepsilon_{p,q+1}\varepsilon_{p,q-1}\varepsilon_{p+1,q+1}\varepsilon_{p-1,q+1} \\ & + \varepsilon_{p+1,q}\varepsilon_{p-1,q}\varepsilon_{p,q}\varepsilon_{p-2,q} + \varepsilon_{p,q+1}\varepsilon_{p,q-1}\varepsilon_{p,q+2}\varepsilon_{p,q} - 1 \\ & + \varepsilon_{p,q+1}\varepsilon_{p,q-1}\varepsilon_{p,q}\varepsilon_{p,q-2} - \varepsilon_{p+1,q}\varepsilon_{p-1,q}\varepsilon_{p,q+1}\varepsilon_{p,q-1} \\ & + \varepsilon_{p+1,q}\varepsilon_{p-1,q}\varepsilon_{p+2,q}\varepsilon_{p,q} + \varepsilon_{p+1,q}\varepsilon_{p-1,q}\varepsilon_{p+1,q+1}\varepsilon_{p+1,q-1},\end{aligned} \tag{44}$$

and

$$\chi_{p,q}^{(4,0)} = \varepsilon_{p-1,q-1}\varepsilon_{p+1,q+1}\,, \quad \chi_{p,q}^{(0,4)} = \varepsilon_{p-1,q+1}\varepsilon_{p+1,q-1}. \tag{45}$$

This procedure may be formulated for $\Omega(m, n)$ with arbitrary ζ as well.

Conjecture 1. *If $\beta > 1$, all seria (42) converge. Solution of (37) is defined uniquely by the distribution of the signs $\boldsymbol{\varepsilon} \equiv \{\varepsilon_{m,n}\}$.*

Note, due to the parity structure of (37), any distribution $\{\varepsilon_{m,n}\}$ is equivalent to $\{\varepsilon'_{m,n} = \varepsilon_{m,n}(\pm)^m(\pm)^n\}$, see Eq. (24).

The homogeneous distribution $\varepsilon_{m,n} = (+)$ is the distinguished one since in this case $c_{m,n}$ are the same for all m, n: $c_{m,n} = c_0$. Expression for c_0 follows from (37) and matches the series form (42),

$$c_0 = \left(\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\beta\Omega(m,n)} \right)^{-1/2}. \tag{46}$$

When $\beta \to 1$, the value of (46) diverges as

$$c_0 \approx \frac{1}{\sqrt{(\beta - 1)\chi}} \tag{47}$$

for some $\chi = \chi(c, \zeta)$. This divergence may be explained by the $\frac{1}{NM}$ term in (31): $\beta = 1 + \frac{f_2}{NM}$ when $N, M \to \infty$, so that asymptotically

$$c_0 = \sqrt{\frac{NM}{f_2\chi}} \sim \sqrt{NM}. \tag{48}$$

The distribution $\varepsilon_{m,n} = (+)$ and $c_{m,n} = c_0 \sim \sqrt{NM}$ is the ground state according to the numerical tests.

If the signs $\varepsilon_{m,n}$ vary for different m, n (even if only one sign is opposite to all the others), we have

Conjecture 2. *The seria (42) with inhomogeneous $\boldsymbol{\varepsilon}$ converge at $\beta = 1$.*

We can explain $c_0 \sim \sqrt{NM}$ in a bit different way. Consider for instance the following distribution of the signs:

$$\varepsilon_{p+m,q+n} = \begin{cases} (+) \text{ if } \Omega(m,n) \le \frac{\pi}{2} V, \\ \\ \text{randomly } (\pm) \text{ if } \Omega(m,n) > \frac{\pi}{2} V \end{cases} \tag{49}$$

In this case a *very rough* estimation gives

$$c_{p,q} \sim \sqrt{V}. \tag{50}$$

Thus the finite-volume domain of positive signs on the infinite lattice is effectively equivalent to finite lattice, and the belt $\Omega(m,n) \sim \frac{\pi}{2} V$ plays the rôle of an effective boundary.

The homogeneous distribution $\varepsilon_{m,n} = (+)$ in a big volume V gives evidently the maximal eigenvalues of the quantum mechanical model, any variation of the signs gives an excitation of the spectrum. A distribution of the signs $\varepsilon_{m_1,n_1} = \varepsilon_{m_2,n_2} = \ldots \varepsilon_{m_k,n_k} = (-)$ with $(m_1, n_1) \ldots (m_k, n_k)$ inside V and with all other $\varepsilon_{m,n} = (+)$ inside V, is a candidate for a k-particles state.

One particle state, $\varepsilon_{m_1,n_1} = (-)$ with all other $\varepsilon_{m,n} = (+)$ inside V, is described asymptotically by two continuous parameters $(\mu, \nu) = (\frac{m_1}{\sqrt{V}}, \frac{n_1}{\sqrt{V}})$. We expect a "dispersion relation" in the form $\frac{\tau_{m,n}}{\sqrt{V}}$ = a smooth function of (μ, ν). The model evidently is gap-less.

The behavior (50) allows one to suggest a candidate for the Hamiltonian of the system:

$$H = -\sum_{m,n} \tau_{m,n}^2 \equiv -\sum_{m,n} c_{m,n}^2 \mathrm{e}^{-\Omega(m,n)}. \tag{51}$$

At the ground state $H \approx -h_0 V$, i.e. one can talk about the density energy $-h_0$ of the ground state, and the spectrum of H describes bound states $-h_0 \leq \frac{H}{V} < 0$.

From the alternative point of view, one may consider the Hamiltonian

$$H' = -H. \tag{52}$$

For this Hamiltonian, the ground state corresponds to a random distribution of the signs—we can say nothing about it. Excitations are the islands of constant signs in the sea of random ones, and its maximal value is described by the finite energy density $+h_0$.

8. DISCUSSION

The main results of this paper are the following. Distribution of the eigenvalues of $t_{p,q}$ near the middle point (P_0, Q_0) (29) of the domain Π (25) is given by

$$\frac{t_{P_0+p,Q_0+q}}{\sqrt{f_0}\mathrm{e}^{\frac{1}{2}NM\mathrm{g}_0(\kappa^2)}} \underset{N,M\to\infty}{=} \mathrm{e}^{-\frac{1}{2}\Omega(p+u_1,q+u_2)} c_{p,q}, \tag{53}$$

notations from Sec. 5. The set of $c_{p,q}$ is the solution of

$$\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\Omega(n,m)} c_{p-m,q-n} c_{p+m,q+n} = 1 \quad \forall\, p, q \in \mathbb{Z}. \tag{54}$$

Any solution of (54) is uniquely defined by the set of signs, $\varepsilon_{p,q}$ = sign of $c_{p,q}$. The ground state distribution corresponding to the homogeneous signs $\varepsilon_{p,q} = (+)$ is given by

$$\frac{t_{P_0+p,Q_0+q}}{\sqrt{f_0}\mathrm{e}^{\frac{1}{2}NM\mathrm{g}_0(\kappa^2)}} \underset{N,M\to\infty}{=} \text{const}\,\sqrt{NM}\,\mathrm{e}^{-\frac{1}{2}\Omega(p+u_1,q+u_2)}. \tag{55}$$

The quadratic form $\Omega(p,q)$ (30) depends on the aspect ratio $\zeta = N/M$, therefore our results are beyond the nested Bethe Ansatz approach.

A number of unsolved problems must be mentioned. Analytical expression for the coefficient f_2 in (31) and related common scale of the ground state distribution are not known. Analytical answer for one-particle distribution of $c_{p,q}$ is not known either. It is also unclear, how a position of single opposite sign of one-particle distribution is related to the momenta of such eigenstate. The list of unsolved problems may be continued.

APPENDIX A. sl_3 FUSION ALGEBRA

Let us demonstrate how the Eqs (13,14) generate the fusion algebra of auxiliary transfer matrices. Choose the particular value $N = 3$. The series (11) may be rewritten as the four-term sum

$$\begin{aligned}
J(x,y) &= \sum_{n=0}^{N}\sum_{m=0}^{M}(-\mathrm{i})^{mn}(-x\sigma^x)^m(-y\sigma^y)^n t_{m,n} \\
&= \left(\sum_{m=0}^{M}(-x\sigma^x)^m t_{m,0}\right) - \left(\sum_{m=0}^{M}(\mathrm{i}x\sigma^x)^m t_{m,1}\right) y\sigma^y \\
&\quad + \left(\sum_{m=0}^{M}(x\sigma^x)^m t_{m,2}\right) y^2 - \left(\sum_{m=0}^{M}(-\mathrm{i}x\sigma^x)^m t_{m,3}\right) y^3\sigma^y \\
&\equiv t_0(x\sigma^x) \;-\; t_1(x\sigma^x)\, y\sigma^y \;+\; t_2(x\sigma^x)\, y^2 \;-\; t_3(x\sigma^x)\, y^3\sigma^y. \qquad (56)
\end{aligned}$$

One may show combinatorially,[9] $t_k(x)$ is the $\mathcal{U}_{q=-1}(\widehat{sl}_N)$ transfer matrix for the Lax operators with the cyclic representation in the quantum space and the fundamental representation π_k in the auxiliary space. It is supposed, π_0 and π_N are the scalar representations, π_1 is the vector representation etc. In $N = 3$ case π_2 is the co-vector representation.

Decomposition of $F(x^2, y^2)$ with respect to y^2 is following ($\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ and $\lambda^3 = x^2$):

$$\begin{aligned}
F(x^2, y^2) &= \prod_{n=0}^{2}\left((1-\lambda\omega^n)^M - y^2(1+\kappa^2\lambda\omega^n)^M\right) \\
&= A(x^2) - B(x^2)y^2 + C(x^2)y^4 - D(x^2)y^6, \qquad (57)
\end{aligned}$$

where

$$A(x^2) \;=\; (1-x^2)^M\,,\quad D(x^2) \;=\; (1+\kappa^6 x^2)^M\,,$$
$$B(x^2) = A(x^2)\left(\left(\frac{1+\kappa^2\lambda}{1-\lambda}\right)^M + \left(\frac{1+\kappa^2\lambda\omega}{1-\lambda\omega}\right)^M + \left(\frac{1+\kappa^2\lambda\omega^2}{1-\lambda\omega^2}\right)^M\right),$$
$$C(x^2) = D(x^2)\left(\left(\frac{1-\lambda}{1+\kappa^2\lambda}\right)^M + \left(\frac{1-\lambda\omega}{1+\kappa^2\lambda\omega}\right)^M + \left(\frac{1-\lambda\omega^2}{1+\kappa^2\lambda\omega^2}\right)^M\right). \quad (58)$$

Equating now (13) in all orders of y^2, one comes at y^0 and y^6 to

$$t_0(x)\,t_0(-x) \;=\; (1-x^2)^M\,,\quad t_3(x)t_3\,(-x) \;=\; (1+\kappa^6x^2)^M. \quad (59)$$

For the generalized chiral Potts model the choice of $U(1)$ charges is prescribed:

$$t_0(x) = (1-x)^M\,,\quad t_3(x) \;=\; (1+\mathrm{i}\kappa^3 x). \quad (60)$$

The orders y^2 and y^4 give

$$\begin{aligned} t_1(x)t_1(-x) \;&=\; t_0(-x)t_2(x)+t_0(x)t_2(-x)+B(x^2),\\ t_2(x)t_2(-x) \;&=\; t_3(-x)t_1(x)+t_3(x)t_1(-x)+C(x^2). \end{aligned} \quad (61)$$

Relations (61) with (60) are exactly the fusion algebra for sl_3.[4,10,11]

APPENDIX B. ASYMPTOTIC OF $F_{P,Q}$

Let us discuss briefly the derivation of (31). Taking into account (22), one may use the saddle point method for the estimation of $F_{P,Q}$. Basically,

$$F_{P,Q} \;=\; \frac{1}{(2\pi \mathrm{i})^2}\oint\oint \frac{dX}{X}\frac{dY}{Y}\frac{F(X,Y)}{X^P Y^Q}. \quad (62)$$

Let

$$\alpha_p \;=\; \frac{P\pi}{M},\quad \alpha_q \;=\; \frac{Q\pi}{N}. \quad (63)$$

Then

$$\log\left(\frac{F(X,Y)}{X^PY^Q}\right) \;\sim\; NM\left(\mathfrak{g}(\lambda,\mu;\kappa^2) - \frac{\alpha_p}{\pi}\log\lambda - \frac{\alpha_q}{\pi}\log\mu\right) \quad (64)$$

It has the extremum (minimum) with respect to λ, μ (κ^2 being fixed) at [2]

$$r_0+r_2 \;=\; \alpha_p,\quad r_0+r_3 \;=\; \alpha_q. \quad (65)$$

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[2] In details, $\lambda\frac{\partial\mathfrak{g}}{\partial\lambda} \;=\; \frac{r_0+r_2}{\pi}$, $\mu\frac{\partial\mathfrak{g}}{\partial\mu} \;=\; \frac{r_0+r_3}{\pi}$, $\kappa^2\frac{\partial\mathfrak{g}}{\partial\kappa^2} \;=\; \frac{r_0}{\pi}$.

The extremum value of $\mathfrak{g}(\lambda, \mu; \kappa^2) - \frac{\alpha_p}{\pi}\log\lambda - \frac{\alpha_q}{\pi}\log\mu$ is

$$g(\alpha_p, \alpha_q; \kappa^2) = \frac{r_0}{\pi}\log\kappa^2 + \sum_{j=0}^{3}\Phi(r_j) \tag{66}$$

where the numbers r_j are to be calculated via

$$\begin{aligned} r_0 &= \pi - \frac{a_1 + a_2 + a_3}{2}, \quad r_1 = \frac{a_2 + a_3 - a_1}{2}, \\ r_2 &= \frac{a_3 + a_1 - a_2}{2}, \quad r_3 = \frac{a_1 + a_2 - a_3}{2}, \end{aligned} \tag{67}$$

and

$$\begin{aligned} a_2 &= \pi - \alpha_p, \quad a_3 = \pi - \alpha_q, \\ a_1 &= \arccos\left(\cos a_2 \cos a_3 + \frac{\kappa^2 - 1}{\kappa^2 + 1}\sin a_2 \sin a_3\right). \end{aligned} \tag{68}$$

The last equality is the solution of $\kappa^2 = \frac{\sin r_0 \sin r_1}{\sin r_2 \sin r_3}$ with respect to a_1. Therefore asymptotically

$$F_{P,Q} = (-)^{P+Q+PQ} \cdot f_0 \cdot \left(1 + \frac{F'}{NM} + \cdots\right) \cdot e^{NMg(\alpha_p,\alpha_q;\kappa^2)}. \tag{69}$$

Function $g(\alpha_q, \alpha_q; \kappa^2)$ has the maximum near $\alpha_p = \alpha_q = \pi - a$, where a is defined by (28), and

$$g(\alpha_p, \alpha_q; \kappa^2) = \mathfrak{g}_0(\kappa^2) - \frac{1 + c^2}{4\pi c}\left(\delta\alpha_p^2 + \delta\alpha_q^2\right) - \frac{1 - c^2}{2\pi c}\delta\alpha_p\delta\alpha_q, \tag{70}$$

where $\mathfrak{g}_0(\kappa^2)$ is given by (32). Let further even integers P_0, Q_0 and real numbers u_1, u_2 are defined by (29). Then

$$\delta\alpha_p = \frac{\pi}{M}(p + u_1), \quad \delta\alpha_q = \frac{\pi}{N}(q + u_2). \tag{71}$$

Therefore, the leading term of (69) is

$$F_{P_0+p,Q_0+q} = (-)^{p+q+pq} \cdot f_0 \cdot e^{NM\mathfrak{g}_0(\kappa^2) - \Omega(p+u_1,q+u_2)}, \tag{72}$$

where the quadratic form is given by (30).

The next order in (69), $F' = f_1 + f_2\Omega(p + u_1, q + u_2)$, is the result of numerical tests.

APPENDIX C. THETA-FUNCTIONS

In the limit $M, N \to \infty$ the polynomial $F(X, Y)$ as well as the eigenstates of $t_{\alpha,\beta}(x, y)$ for periodical distribution of the signs $\varepsilon_{m,n}$ become the theta-functions. In particular, eqs. (34) may be re-written in a theta-functions-like form:

$$\left(\sum x^{2m} y^{2n} \tau_{2m,2n}\right)^2 - \left(\sum (-)^n x^{2m+1} y^{2n} \tau_{2m+1,2n}\right)^2$$
$$- \left(\sum (-)^m x^{2m} y^{2n+1} \tau_{2m,2n+1}\right)^2 - \left(\sum (-)^{n+m} x^{2m+1} y^{2n+1} \tau_{2m+1,2n+1}\right)^2$$
$$= \sum_{p,q} (-)^{p+q+pq} \mathrm{e}^{-\Omega(p+u_1,q+u_2)} x^{2p} y^{2q} \tag{73}$$

Let us re-define $x = \mathrm{e}^{\mathrm{i}\pi z_1}$ and $y = \mathrm{e}^{\mathrm{i}\pi z_2}$. Then the theta-function-like seria

$$\tau_{\alpha,\beta}(z_1, z_2) = \sum_{m,n\in\mathbb{Z}} (-)^{\alpha n+\beta m} \tau_{2m+\alpha,2n+\beta} \mathrm{e}^{\mathrm{i}\pi(2m+\alpha)z_1+\mathrm{i}}\pi(2n+\beta)nz_2. \tag{74}$$

stand for the transfer matrices.

It is helpful to discuss some properties of theta-functions. Let

$$\Theta^{(\beta)}_{u_1,u_2}(z_1, z_2) = \sum_{p,q} \mathrm{e}^{-\beta\Omega(p+u_1,q+u_2)+2\pi \mathrm{i} p z_1+2\pi \mathrm{i} q z_2} \tag{75}$$

for our particular quadratic form Ω (30). It has the general Jacobi transform property:

$$\Theta^{(\beta)}_{u_1,u_2}(z_1, z_2) = \frac{2}{\beta} \mathrm{e}^{-2\pi \mathrm{i}(z_1 u_1+z_2 u_2)} \Theta^{(4/\beta)}_{z_2,-z_1}(-u_2, u_1). \tag{76}$$

The other θ-function, related to F, is

$$F_{u_1,u_2}(z_1, z_2) = \sum_{p,q} (-)^{p+q+pq} \mathrm{e}^{-\Omega(p+u_1,q+u_2)+2\pi \mathrm{i} z_1 p+2\pi \mathrm{i} z_2 q}. \tag{77}$$

One can easily see,

$$F_{u_1,u_2}\left(z_1, z_2\right) = \frac{1}{2}\left(\Theta^{(1)}_{u_1,u_2}\left(z_1+\frac{1}{2}, z_2+\frac{1}{2}\right)\right.$$
$$\left. + \Theta^{(1)}_{u_1,u_2}\left(z_1+\frac{1}{2}, z_2\right) + \Theta^{(1)}_{u_1,u_2}\left(z_1, z_2+\frac{1}{2}\right) - \Theta^{(1)}_{u_1,u_2}(z_1, z_2)\right)$$
$$= \left(2\Theta^{(4)}_{u_1/2,u_2/2}(2z_1, 2z_2) - \Theta^{(1)}_{u_1,u_2}(z_1, z_2)\right). \tag{78}$$

For the case $u_1 = u_2 = 0$, the polynomial identity $F_{2N,2M}(x^2, y^2) = F_{N,M}(x, y)F_{N,M}(-x, y)F_{N,M}(x, -y)F_{N,M}(-x, -y)$ provides

$$f_0 F_{0,0}(z_1, z_2) = f_0^4 F_{0,0}\left(\frac{z_1}{2}, \frac{z_2}{2}\right) F_{0,0}\left(\frac{z_1+1}{2}, \frac{z_2}{2}\right) \times F_{0,0}\left(\frac{z_1}{2}, \frac{z_2+1}{2}\right) F_{0,0}\left(\frac{z_1+1}{2}, \frac{z_2+1}{2}\right). \tag{79}$$

The limit $z_1, z_2 \to 0$ gives f_0 for (31):

$$f_0 = \sqrt[3]{\frac{4}{F_{0,0}\left(\frac{1}{2}, 0\right) F_{0,0}\left(0, \frac{1}{2}\right) F_{0,0}\left(\frac{1}{2}, \frac{1}{2}\right)}}. \tag{80}$$

As well, the value of χ for (47) follows from

$$\sum_{m,n} (-)^{m+n+mn} \mathrm{e}^{-\beta\Omega(m,n)} = F_{0,0}^{(\beta)}(0, 0) = \frac{1}{\beta}\Theta_{0,0}^{(1/\beta)} - \Theta_{0,0}^{(\beta)} \approx (1-\beta)\chi \tag{81}$$

at $\beta \to 1$ with $\chi = \Theta_{0,0}^{(1)} + 2\frac{\partial\Theta_{0,0}^{(\beta)}}{\partial\beta}|_{\beta=1}$.

APPENDIX D. EXAMPLES OF PERIODICAL DISTRIBUTION

Here we give an example is a periodical distribution of the signs. Let

$$\varepsilon_{2m+\alpha,2n+\beta} = \varepsilon_{\alpha,\beta}\, \mathrm{e}^{\mathrm{i}\pi(um+vn)} \tag{82}$$

with $u, v = 0$ or 1. Periodicity of $\varepsilon_{m,n}$ provides the periodicity of the series expansions (42), and therefore

$$c_{2m+\alpha,2n+\beta} = \varepsilon_{2m+\alpha,2n+\beta}\, c_{\alpha,\beta}. \tag{83}$$

Equation (37) gives

$$c_{\alpha,\beta}^2 \Theta_{0,0}^{(4\beta)} - c_{1-\alpha,\beta}^2 \mathrm{e}^{\mathrm{i}\pi u} \Theta_{\frac{1}{2},0}^{(4\beta)} - c_{\alpha,1-\beta}^2 \mathrm{e}^{\mathrm{i}\pi v} \Theta_{0,\frac{1}{2}}^{(4\beta)} - c_{1-\alpha,1-\beta}^2 \mathrm{e}^{\mathrm{i}\pi(u+v)} \Theta_{\frac{1}{2},\frac{1}{2}}^{(4\beta)} = 1 \tag{84}$$

for all four choices of (α, β), its solution is $c_{0,0}^2 = c_{1,0}^2 = c_{0,1}^2 = c_{1,1}^2$ (it follows from the careful analysis of the structure of $\varepsilon_{m,n}$-products in (42)), so that

$$c_{2m+\alpha,2n+\beta} = \varepsilon_{\alpha,\beta}\, \mathrm{e}^{\mathrm{i}\pi(um+vn)} \times \left(\Theta_{0,0}^{(4\beta)} - \mathrm{e}^{\mathrm{i}\pi u} \Theta_{\frac{1}{2},0}^{(4\beta)} - \mathrm{e}^{\mathrm{i}\pi v} \Theta_{0,\frac{1}{2}}^{(4\beta)} - \mathrm{e}^{\mathrm{i}\pi(u+v)} \Theta_{\frac{1}{2},\frac{1}{2}}^{(4\beta)}\right)^{-1/2}. \tag{85}$$

APPENDIX E. TRANSFER MATRIX OF ZAMOLODCHIKOV—BAZHANOV—BAXTER MODEL

In the last section we would like to describe the relation between (22) and Baxter's free energy for Zamolodchikov's model. We will refer to,[13] where the inhomogeneous model was considered and *divisor* parameterization was used. Equations (231) in Ref. 13 look like

$$J(X) \cdot \mathbf{T} \;=\; \mathbf{T} \cdot J(X') \;=\; 0. \tag{86}$$

Here $J(X)$ and $J(X')$ are generating functions (10), operator $\mathbf{T}$ is a modified transfer matrix for Zamolodchikov-Bazhanov-Baxter model (in general, the Pauli matrices may be replaced by the Weyl algebra generators at root of unity). It follows from (86), $\mathbf{T}$ up to a normalization is the product of algebraic supplements of $J(X)$ and $J(X')$.

In our particular case, $J(X)$, $J(X')$ and $\mathbf{T}$ after the quasi-diagonalization are 2×2 matrices (in the basis of the Pauli matrices). Transfer-matrix of Zamolodchikov's model $T(\theta_1, \theta_2, \theta_3)$, mentioned in the Introduction, is the trace of $\mathbf{T}$:

$$T \;=\; \mathrm{Trace}_{2\times 2}\mathbf{T}. \tag{87}$$

Generating functions $J(X)$ and $J(X')$ stand for $J(\lambda(X)^{N/2}, \mu(X)^{M/2}; \kappa^2)$ and $J(\lambda(X')^{N/2}, \mu(X')^{M/2}; \kappa^2)$ in the present notations, where

$$\kappa^2 \;=\; \tan^2\frac{\theta_1}{2} \;=\; \frac{\sin\beta_2 \sin\beta_3}{\sin\beta_0 \sin\beta_1} \tag{88}$$

is the κ-parameter in both $J(X)$ and $J(X')$, and explicit evaluations for λ and μ from[13] to the terms of linear excesses β_j give

$$\lambda(X) \;=\; \mathrm{e}^{-\mathrm{i}(\beta_1+\beta_2)}\frac{\sin\beta_0}{\sin\beta_3}\,, \quad \mu(X) \;=\; \mathrm{e}^{\mathrm{i}(\beta_0+\beta_2)}\frac{\sin\beta_1}{\sin\beta_3}\,, \tag{89}$$

and

$$\lambda(X') \;=\; \mathrm{e}^{\mathrm{i}(\beta_0+\beta_3)}\frac{\sin\beta_1}{\sin\beta_2}\,, \quad \mu(X') \;=\; \mathrm{e}^{-\mathrm{i}(\beta_1+\beta_3)}\frac{\sin\beta_0}{\sin\beta_2}. \tag{90}$$

It gives us the identification $\{r_j\} = \{$a permutation of $\beta_j\}$ and relates (22) to Baxter's answer for the partition function per site k:

$$\log k \;=\; \text{normalization} \;+\; \sum_{j=0}^{3}\left(\frac{\beta_j}{2\pi}\log 2\sin\beta_j + \Phi(\beta_j)\right). \tag{91}$$

The reader may see the discrepancy, $\frac{\beta_j}{2\pi}\log 2\sin\beta_j$ in (91) and $\frac{\beta_j}{\pi}\log 2\sin\beta_j$ in (22), it means that the normalization is not trivial—it comes from a certain variational principle.

9. ACKNOWLEDGEMENTS

This work was supported by the Australian Research Council. The author should like to thank V. Bazhanov, V. Mangazeev, M. Batchelor, X-W. Guan and all the Mathematical Physics group of the Department of Theoretical Physics of RSPhysSE for fruitful discussions.

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